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Global and local stability analysis in a nonlinear discrete-time population model

Özlem Ak Gümüş*

*Correspondence:
akgumus@adiyaman.edu.tr
Department of Mathematics,
Faculty of Arts and Sciences,
Adiyaman University, Adiyaman,
02040, Turkey

Abstract

In this article, we investigate the local and global stability conditions of equilibrium points of discrete-time dynamic model with and without Allee effect. We conclude that the Allee effect decreases both the local stability and the global stability of equilibrium points of the population dynamic model. The results are confirmed with a numerical simulation.

MSC: 39A10; 39A30

Keywords: global stability; local stability; Allee effect; population model; equilibrium point; discrete-time models

1 Introduction

It is well known that the Allee effect plays an important role in the stability analysis of equilibrium points of a population dynamics model (see, for instance, [1–13]). The Allee effect, first introduced by Allee [14] in 1931, represents a negative density dependence when the population growth rate is reduced at low population size. It may be due to a number of sources including difficulties in finding mates, inbreeding depression, food exploitation, predator avoidance of defense, and social dysfunction at small population sizes.

Real-world problems can be solved by examining the models created via differential or difference equations. The population models in ecology and biology are generally confirmed by such equations (see, for instance, [9, 14–16]). To examine the dynamics of the population requires stability analysis. For local asymptotic stability, solutions must approach an equilibrium point under initial conditions close to the equilibrium point. In global asymptotic stability, solutions must approach to an equilibrium point under all initial conditions. The global asymptotic stability has been investigated in (see, for instance, [17–19]). Since a globally attractive equilibrium point is locally attractive, a globally asymptotically stable equilibrium point is locally asymptotically stable. Also, if the function is continuous, global asymptotic stability and global attractiveness are equivalent. In this study, we are not concerned with global attractiveness, which is derived with the help of solutions of the equation. The purpose of this paper is to investigate the local and global stability of an equilibrium point analytically with and without Allee effect and to compare the stability of these models. Therefore, this article exposes the impact of the Allee effect on the local and global stability of an equilibrium point in a nonlinear discrete-time population model. The positive equilibrium point of the model which is subject to an Allee effect can become either one of destabilization (see, for instance, [11, 13]) or of stabilization

(see, for instance, [1, 5, 7, 10]). Namely, the local stability of a positive equilibrium point can be changed from the stable case to an unstable case or vice versa. It is also possible that even if the model is stable at an equilibrium point, to reach its equilibrium point may take a much longer time. This case has been referred to in the statement that the 'Allee effect decreases the local stability of the equilibrium point' (see, for instance, [6, 8, 12]).

In [20], a linear population model depending on birth and death rates was introduced:

$$N_{t+1} = rN_t.$$

Here, N_t is the population density at time t , and r ($r = 1 + b - d$) is a growth rate. Unfortunately, this model neglects many important aspects of biological reality, since most of the parameters connected to the interactions among individuals are overlooked. Thus, nonlinear population models offer a more realistic approach than linear population models.

In this paper, we consider the following nonlinear discrete-time population model:

$$N_{t+1} = rN_t + f(N_t) = F(r, N_t), \quad (1)$$

where $f(N_t)$ represents interactions (competitions) among mature individuals, and r is the growth rate such that $r \in (0, 1)$. The following assumptions are imposed on the function f :

- (i) $f'(N_t) < 0$ for $N_t \in [0, \infty)$; i.e., as the density increases, f decreases continuously. Biologically speaking, social dysfunction never increases as the population size increases.
- (ii) $f(0)$ has a finite positive value.

If the population model of (1) is subject to the Allee effect, the following nonlinear population model is used:

$$N_{t+1} = r^* \alpha(N_t) N_t + f(N_t) = F_\alpha(r^*, \alpha, N_t). \quad (2)$$

The function f satisfies the conditions (i) and (ii). Note that since $r^* > 0$ is related to the normalized *per capita* growth rate given by $r/\alpha(N^*)$, (1) and (2) have the same equilibrium point. Here, $\alpha := \alpha(N_t)$ is the Allee function at time t . The following assumptions on the Allee function are derived from biological facts:

- (iii) If there are no partners, there is no reproduction. Mathematically speaking, the Allee function is zero when the population density is zero.
- (iv) The Allee effect increases as the density increases. Mathematically speaking, the derivatives of the Allee function are always positive for all positive values.
- (v) The Allee effect disappears at high densities. Namely, the limit of the Allee function approaches 1 as the population size increases.

The remainder of the article is organized as follows: Section 2 is concerned with a local stability analysis of the equilibrium points of (1) with and without Allee effect. In Section 3, we give a global stability analysis of the equilibrium points of (1) with and without the Allee effect. In Section 4, we present numerical simulations to corroborate our results. The final section is devoted to conclusions and remarks.

2 Local stability analysis of (1) with and without Allee effect

We will begin by reviewing some definitions and theorems (see, for instance, [15]) which will be useful in our study of the stability analysis of (1).

Definition 1 N^* is defined as an equilibrium point of (1) if the following equality holds:

$$N^* = F(N^*).$$

Theorem 2 Let N^* be a positive equilibrium point of (1). Assume that F' is continuous on an open interval I containing an equilibrium point. Then N^* is locally asymptotically stable if

$$|F'(N^*)| < 1,$$

and unstable if

$$|F'(N^*)| > 1.$$

Definition 3 $m > 1$ is defined as the period for (1) if

$$F^m(N^*) = N^* \quad \text{and} \quad F^i(N^*) \neq N^*, \quad \text{for } i = 1, 2, \dots, m-1.$$

We will examine the local stability of the equilibrium point of (1) with and without the Allee effect. Assume that (1) and (2) have a unique positive equilibrium point N^* . Namely, there is no equilibrium point except N^* such that $F(F(N^*)) \neq N^*$.

We then obtain the following theorem.

Theorem 4 Let N^* be a positive equilibrium point of (1) with respect to r . Then N^* is locally stable if the inequality

$$f'(N^*) > -1 - r \tag{3}$$

holds.

Proof In light of (i)-(ii), we can say that F' is a continuous function. The linearized form of (1) in a neighborhood of N^* is given by

$$u_{t+1} = F'(N^*)u_t, \quad F'(N^*) = r + f'(N^*),$$

such that $u_t = N_t - N^*$. By applying Theorem 2, we find that

$$-1 - r < f'(N^*) < 1 - r.$$

Thus the inequality (3) is proved for $f' < 0$. □

Theorem 5 Let N^* be a positive equilibrium point of (2) by the Allee effect at time t with respect to r^* . Then N^* is locally stable if the inequality

$$-r - rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} < f'(N^*) < 1 - r - rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} \quad (4)$$

holds.

Proof From (i)-(v), we can say that F'_α is a continuous function. The linearized form of (2) in a neighborhood of N^* is given by

$$u_{t+1} = F'_\alpha(N^*)u_t$$

such that $u_t = N_t - N^*$. By applying Theorem 2, we have

$$\begin{aligned} |F'_\alpha(N^*)| < 1 &\Rightarrow |r^*\alpha'(N^*)N^* + r^*\alpha(N^*) + f'(N^*)| < 1 \\ &\Rightarrow -1 < rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + f'(N^*) + r < 1 \\ &\Rightarrow -1 - r < rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + f'(N^*) < 1 - r. \end{aligned} \quad (5)$$

Here, $r = r^*\alpha(N^*)$ and $\alpha(N^*) < 1$. The second derivative of the function f is

$$f''(x) = \lim_{h \rightarrow 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

Since f decreases, it is written

$$\begin{aligned} x+h < x+2h &\Rightarrow f(x+h) > f(x+2h), \\ x < x+2h &\Rightarrow f(x) > f(x+2h). \end{aligned}$$

From this, we get the following inequality:

$$f(x+h) + f(x) > 2f(x+2h),$$

so that $f''(x) > 0$. Let us consider the first and second derivatives of the function F_α as follows:

$$\begin{aligned} F'_\alpha &= r^*\alpha'(N^*)N^* + r^*\alpha(N^*) + f'(N^*), \\ F''_\alpha &= r^*\alpha''(N^*)N^* + 2r^*\alpha'(N^*) + f''(N^*), \end{aligned}$$

such that $|\alpha''(N^*)| < \alpha'(N^*)$ in the case of $\alpha''(N^*) < 0$. Consequently, we get $F''_\alpha > 0$ so that F_α is a convex function. Also it is seen that $F_\alpha \rightarrow \infty$ for $h \rightarrow 0$. Since F_α is an increasing function in $[N^*, \infty)$, we can write $rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + f'(N^*) > -r$. We then obtain the following:

$$-r < rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + f'(N^*) < 1 - r.$$

If the last inequality and inequality (5) are considered, then (4) is confirmed easily. \square

Corollary 6 *It can be seen from (3) and (4) that the Allee effect decreases the local stability of an equilibrium point of (1). Namely, it is easy to see that the local stability of an equilibrium point of (1) is stronger than the local stability of an equilibrium point of (2).*

3 Global stability analysis of (1) with and without Allee effect

In this section, we will present the global stability analysis of an equilibrium point of (1) with and without Allee effect. We shall require the following global stability theorem and definition (see, for instance, [15]) for (1).

Definition 7 N^* is said to be globally asymptotically stable if it is globally attractive and locally stable.

Theorem 8 *Let the function F at (1) be continuous such that $F : [0, p) \rightarrow [0, p)$, $0 < p \leq \infty$, if $0 < F(N) < N$ for all $N \in (0, p)$, then the origin is globally asymptotically stable.*

We then obtain the following theorem.

Theorem 9 *The equilibrium point N^* is globally asymptotically stable if*

$$f'(N^*) > -r. \quad (6)$$

Proof The function of F at (1) is continuous such that $F : [0, p) \rightarrow [0, p)$. The linearized equation about N^* is

$$u_{T+1} = [r + f'(N^*)]u_T$$

such that $u_T = N_t - N^*$. Let us take $F_l = r + f'(N^*)$. Since $f'(N^*) < 0$ from condition (i) with $0 < r < 1$, we can write

$$\begin{aligned} r + f'(N^*) - r &< 0 \\ \Rightarrow (r + f'(N^*) - r)u &< 0 \\ \Rightarrow (r + f'(N^*))u &< ru \\ \Rightarrow (r + f'(N^*))u &< u \\ \Rightarrow F_l(u) &< u. \end{aligned}$$

Now, we must satisfy $0 < F_l(u)$ for the global stability of the zero equilibrium point (origin) as follows:

$$\begin{aligned} (r + f'(N^*))u &> 0 \\ \Rightarrow r + f'(N^*) &> 0 \\ \Rightarrow f'(N^*) &> -r. \end{aligned}$$

It follows that $0 < F_l(u) < u$. By Theorem 8, the origin is globally asymptotically stable. Therefore, the equilibrium point N^* of (1) is globally asymptotically stable. \square

3.1 The Allee effect at time t

We will now consider the following nonlinear discrete-time dynamical system with Allee effect at time t , given by (2):

$$N_{t+1} = r^* N_t \alpha(N_t) + f(N_t) = F_\alpha(r^*, \alpha, N_t). \quad (7)$$

According to the information, F_α has a unique positive equilibrium point $N^* \in (0, p)$.

We then obtain the following theorem.

Theorem 10 *The equilibrium point N^* of (7) is globally asymptotically stable if*

$$-r \left[1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] < f'(N^*) \quad \text{and} \quad r \left[1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] < 1. \quad (8)$$

Proof The function F_α at (7) is continuous function such that $F_\alpha : [0, p) \rightarrow [0, p)$. The linearized equation about N^* is

$$u_{T+1} = \left[f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r \right] u_T$$

such that $u_T = N_t - N^*$. Since $f'(N^*) < 0$ from condition (i), we can write

$$\begin{aligned} f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r &< rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r \\ \Rightarrow \left[f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r \right] u &< r \left[N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + 1 \right] u \end{aligned}$$

such that $rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r > 0$. If $r \left[1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] < 1$, then

$$\left[f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} + r \right] u < u.$$

If $-r < \left[f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right]$, we have

$$\begin{aligned} -r &< \left[f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] \\ \Rightarrow 0 &< \left[r + f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] \\ \Rightarrow 0 &< \left[r + f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] u. \end{aligned}$$

Let us take $F_{l,a} = r + f'(N^*) + rN^* \frac{\alpha'(N^*)}{\alpha(N^*)}$, provided that $0 < F_{l,a}(u) < u$. It is clear from Theorem 8 that the origin is globally asymptotically stable. Then N^* is globally asymptotically stable. Note that $F_{l,a} > 0$ is always true from Theorem 5. Namely, $-r \left[1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] < f'(N^*)$ always. Since this inequality is related with $\left[1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right] < 1$, we must take this inequality as stability conditions. \square

Corollary 11 *The Allee effect at time t decreases the global stability of an equilibrium point. (If (6) and (8) are considered, it can easily be seen.)*

Corollary 12 *It is clear from (1) that N^* is locally stable but not globally stable if the inequality*

$$-1 - r < f(N^*) < -r$$

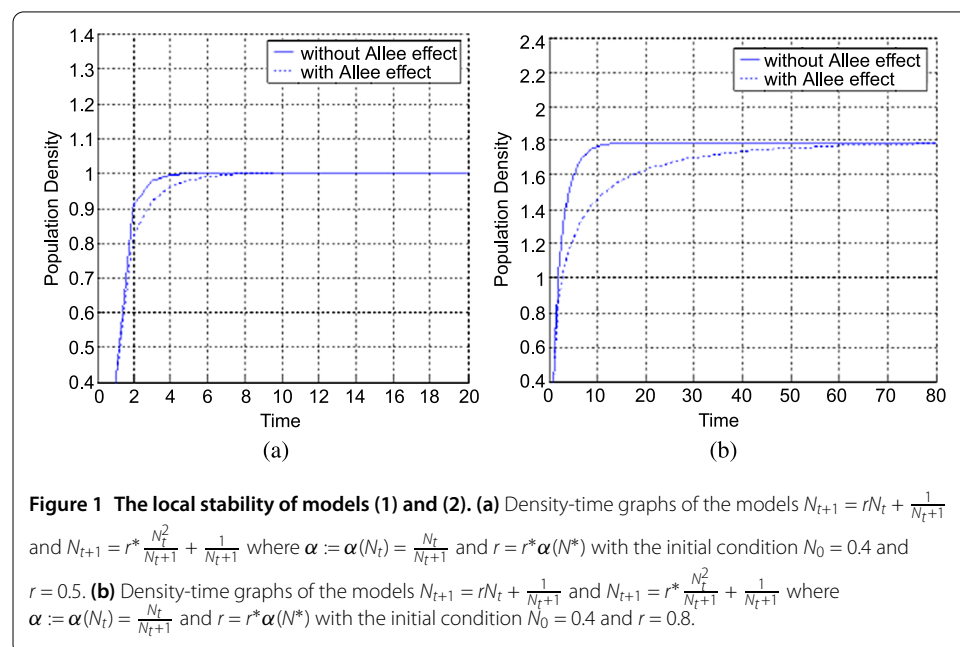
holds. (The proof is clear from Theorems 4 and 9.)

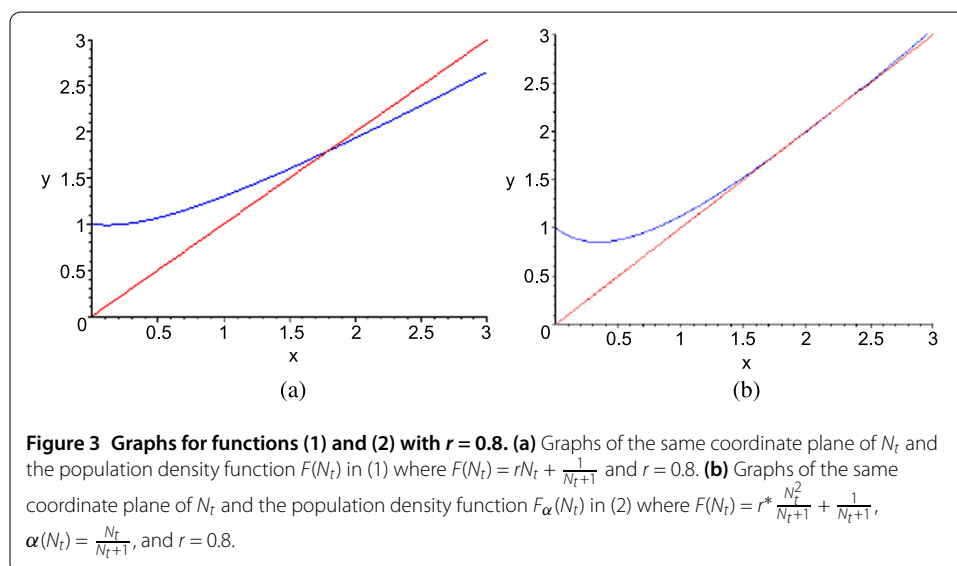
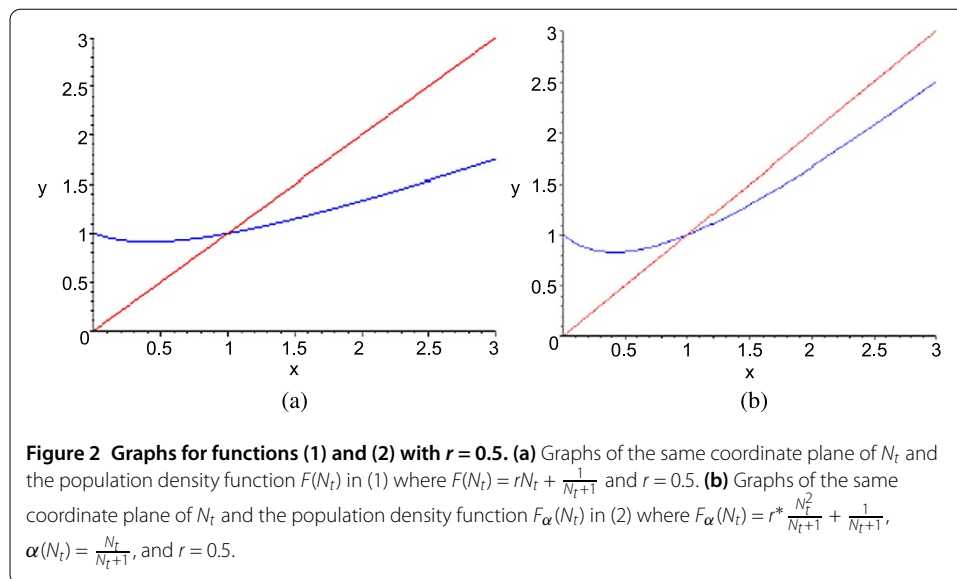
4 Numerical simulations

In this section, we corroborate our results numerically by using Maple and Matlab. As an example, we consider

$$f(N_t) := \frac{1}{N_t + 1} \quad \text{and} \quad \alpha := \alpha(N_t) = \frac{N_t}{N_t + \alpha}.$$

Figure 1 shows a graph which demonstrates the dynamics of the population model based on these functions with and without the Allee effect with respect to time for different values of r . Figures 2 and 3 show the graphs of the same coordinate plane of N_t and the population density functions $F(N_t)$ and $F_a(N_t)$ for $r = 0.5$ and $r = 0.8$, respectively. So, we confirm that the Allee effect decreases the local stability and global stability of the equilibrium point of the population model. It is clear that (1) is not globally stable at each locally stable point.





5 Conclusions and remarks

This paper focused on the global and local stability analysis of the first-order discrete population models with and without the Allee effect, defined by (2) and (1), respectively. Firstly, local asymptotic stability conditions were investigated for the equilibrium points of both models. Secondly, the global stability of the equilibrium points of the models was also evaluated. Finally, we compared the global and local stability of the equilibrium points of these two models. Consequently, the Allee effect decreases the local stability and the global stability of the equilibrium points of (1).

Competing interests

The author declares to have no competing interests.

Author's contributions

The author read and approved the final manuscript.

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